Boxicity of Leaf Powers

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Abstract. The boxicity of a graph G, denoted as $\text{box}(G)$ is defined as the minimum integer t such that G is an intersection graph of axis-parallel t-dimensional boxes. A graph G is a k-leaf power if there exists a tree T such that the leaves of the tree correspond to the vertices of G and two vertices in G are adjacent if and only if their corresponding leaves in T are at a distance of at most k. Leaf powers are a subclass of strongly chordal graphs and are used in the construction of phylogenetic trees in evolutionary biology. We show that for a k-leaf power G, $\text{box}(G) \leq k-1$. We also show the tightness of this bound by constructing a k-leaf power with boxicity equal to $k-1$. This result implies that there exists strongly chordal graphs with arbitrarily high boxicity which is somewhat counterintuitive.

Key words: Boxicity, leaf powers, tree powers, strongly chordal graphs, interval graphs.

1 Introduction

An *axis-parallel k-dimesional box*, or *k-box* in short, is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each R_i is an interval of the form $[a_i, b_i]$ on the real line. A 1-box is thus just a closed interval on the real line and a 2-box a rectangle in $\mathbb{R}^2$ with its sides parallel to the axes. A graph $G(V,E)$ is said to be an *intersection graph* of k-boxes if there is a mapping f that maps the vertices of G to k-boxes such that for any two vertices $u, v \in V$, $(u,v) \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. Then, f is called a *k-box representation* of G. Thus interval graphs are exactly the intersection graphs of 1-boxes. Clearly, a graph that is an intersection graph of k-boxes is also an intersection graph of j boxes for any $j \geq k$. The *boxicity* of a graph G, denoted as $\text{box}(G)$, is the minimum integer k such that G is an intersection graph of k-boxes.

Roberts[19] gave an upper bound of $n/2$ for the boxicity of any graph on n vertices and showed that the complete $n/2$-partite graph with 2 vertices in each part achieves this boxicity. Boxicity has also been shown to have upper bounds in terms of other graph parameters such as the maximum degree and the treewidth[7]. It was shown in [5] that for any graph G on n vertices and having maximum degree Δ, $\text{box}(G) \leq \lceil (\Delta+2)\ln n \rceil$. The same authors showed in [4] that $\text{box}(G) \leq 2\Delta^2$. This result shows that the boxicity of any graph with bounded degree is bounded no matter how large the graph is.

The boxicity of several special classes of graphs have also been studied. Scheinerman [20] showed that outerplanar graphs have boxicity at most 2 while Thomassen [21] showed that every planar graph has boxicity at most 3. The boxicity of series-parallel graphs was studied in [1] and that of Halin graphs in [6].

Graphs which have no induced cycle of length at least 4 are called chordal graphs. Chordal graphs in general can have unbounded boxicity since there are split graphs (a subclass of chordal graphs) that have arbitrarily high boxicity [8]. Strongly chordal graphs are chordal graphs with no induced trampoline[12] (trampolines are also known as "sun graphs"). Several other characterizations of strongly chordal graphs can be found in [16], [15], [9] and [10].

1.1 Leaf powers

A graph G is said to be a *k-leaf power* if there exists a tree T and a correspondence between the vertices of G and the leaves of T such that two vertices in G are adjacent if and only if the distance between their corresponding leaves in T is at most k. The tree T is then called a *k-leaf root* of G. k-leaf powers were introduced by Nishimura et. al.[17] in relation to the phylogenetic reconstruction problem in computational biology. Characterization of 3-leaf powers and a linear time algorithm for their recognition was given in [2]. Clearly, leaf powers are induced subgraphs of the powers of trees. Now, since trees are strongly chordal and any power of any strongly chordal graph is also strongly chordal (as shown in [18] and [9]), leaf powers are also strongly chordal graphs.

1.2 Our results

We show that the boxicity of any k-leaf power is at most $k-1$ and also demonstrate the tightness of this bound by constructing k-leaf powers that have boxicity equal to $k-1$, for $k>1$. The tightness result implies that strongly chordal graphs can have arbitrary boxicity. This is somewhat surprising because when we study the boxicity of strongly chordal graphs, it is tempting to conjecture that boxicity of any strongly chordal graph may be bounded above by some constant and small examples seem to confirm this conjecture. A subclass of strongly chordal graphs, called *strictly chordal graphs*, is studied in [13]. The graphs in this class are shown to be 4-leaf powers in [3]. Therefore strictly chordal graphs have boxicity at most 3.

2 Definitions and notations

We study only simple, undirected and finite graphs. Let $G(V,E)$ denote a graph G on vertex set $V(G)$ and edge set $E(G)$. For any graph G, the number of edges in it is denoted by $||G||$. Thus, if P is a path, $||P||$ denotes the length of the path. If T is a tree that contains vertices u and v, then uTv denotes the unique path in T. For $u, v \in V(T)$, let $d_T(u,v) := ||uTv||$ be the distance between u and v

in T. The k-th power of a graph G, denoted by G^k, is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{(u, v) \mid u, v \in V(G) \text{ and } d_G(u, v) \leq k\}$.

A set X of three independent vertices in a graph G is said to form an *asteroidal triple* if for any $u \in X$, there exists a path P between the two vertices in $X - \{u\}$ such that $N(u) \cap V(P) = \emptyset$ where $V(P)$ denotes the set of vertices in P. A graph is said to be *asteroidal triple-free*, or AT-free in short, if it does not contain any asteroidal triple.

Lemma 1 (Lekkerkerker and Boland[14]). *A graph is an interval graph if and only if it is chordal and asteroidal triple-free.*

If $G_1, \ldots, G_k$ are graphs on the same vertex set V, we denote by $G_1 \cap \cdots \cap G_k$ the graph on V with edge set $E(G_1) \cap \cdots \cap E(G_k)$.

Lemma 2 (Roberts[19]). *For any graph G,* $\mathrm{box}(G) \leq k$ *if and only if there exists a collection of k interval graphs $I_1, \ldots, I_k$ such that $G = \bigcap_{i=1}^{k} I_i$.*

A *critical clique* in a graph is a maximal clique such that every vertex in the clique has the same neighbourhood in G. The *critical clique graph* of a graph G, denoted as $CC(G)$, is a graph in which there is a vertex for every critical clique of G and two vertices in $CC(G)$ are adjacent if and only if the critical cliques corresponding to them in G together induce a clique in G.

Lemma 3. *For any graph G,* $\mathrm{box}(G) = \mathrm{box}(CC(G))$.

Proof. Since $CC(G)$ is an induced subgraph of G, $\mathrm{box}(CC(G)) \leq \mathrm{box}(G)$. Now suppose that u is a vertex in G and G' is the graph formed by adding a vertex u' to $V(G)$ such that $V(G') = V(G) \cup \{u'\}$ and $E(G') = E(G) \cup (u, u') \cup \{(x, u') \mid (x, u) \in E(G)\}$. Since a k-box representation f' for G' can be obtained from a k-box representation f for G by extending f to f' by defining $f'(u) = f(u)$, $\mathrm{box}(G') \leq \mathrm{box}(G)$. Now since any graph G can be obtained from $CC(G)$ by repeatedly performing this operation, $\mathrm{box}(G) \leq \mathrm{box}(CC(G))$. □

A graph G is a k*-Steiner power* if there exists a tree T, called the k*-Steiner root* of G with $|V(T)| \geq |V(G)|$, and an injective map f from $V(G)$ to $V(T)$ such that for $u, v \in V(G)$, $(u, v) \in E(G) \Leftrightarrow d_T(f(u), f(v)) \leq k$. Note that G is induced in T^k by the vertices in $f(V(G))$.

Lemma 4 (Dom et al.[11]). *For $k \geq 3$, a graph G is a k-leaf power if and only if $CC(G)$ is a $(k-2)$-Steiner power.*

We first study the boxicity of tree powers and then deduce our results for leaf powers as corollaries.

3 Boxicity of tree powers

3.1 An upper bound

We show that if T is any tree, boxicity of T^k is at most $k + 1$.

Let T be any tree. Fix some non-leaf vertex r to be the root of the tree. Let m be the number of leaves of the tree T. Let $l_1, \dots, l_m$ be the leaves of T in the order in which they appear in some depth-first traversal of T starting from r.

Define the *ancestor* relation on $V(T)$ as follows: a vertex u is said to be an ancestor of a vertex v, denoted as $u \preceq v$, if $u \in rTv$. Similarly, we use the notation $u \succeq v$ to denote the fact that u is a *descendant* of v, or in other words, v is an ancestor of u.

For any vertex $u \neq r$, let $p(u)$ be the *parent* of u, i.e. the only ancestor of u adjacent to it. Let $p(r) = r$. For any vertex u, we define $p^0(u) = u$, $p^1(u) = p(u)$ and $p^i(u) = p(p^{i-1}(u))$, for $i \geq 2$.

For any vertex u, define $L(u)$ to be the set of indices of leaves of T that are descendants of u, i.e., $L(u) = \{i \mid l_i \succeq u\}$. Define $s(u) = \min\{L(u)\}$ and $t(u) = \max\{L(u)\}$.

Lemma 5. *If $u \preceq v$, then $s(u) \leq s(v) \leq t(v) \leq t(u)$.*

Proof. $u \preceq v \Rightarrow L(v) \subseteq L(u)$. Hence the lemma follows. □

Lemma 6. *If $u \not\preceq v$ and $v \not\preceq u$, then either $s(u) \leq t(u) < s(v)$ or $s(v) \leq t(v) < s(u)$.*

Proof. Since the leaves were ordered in the sequence in which they appear in a depth-first traversal of T from r, for any vertex u, the leaves in $L(u)$ appear consecutively in the ordering $l_1, \dots, l_m$. Since $u \not\preceq v$ and $v \not\preceq u$, $L(u) \cap L(v) = \emptyset$. This proves the lemma. □

In order to show that $\mathrm{box}(T^k) \leq k+1$, we construct $k+1$ interval graphs $I', I_0, \dots, I_{k-1}$ such that $T^k = I' \cap I_0 \cap \dots \cap I_{k-1}$. These interval graphs are constructed as follows.

Construction of I_i, $0 \leq i \leq k-1$:
Let $f_i(u)$ be the interval assigned to vertex u in I_i, i.e., $V(I_i) = V(T)$ and $E(I_i) = \{(u, v) \mid f_i(u) \cap f_i(v) \neq \emptyset\}$. f_i is defined as:

$$f_i(u) = [s(p^i(u)), t(p^{k-1-i}(u))]$$

Note that from Lemma 5, $s(p^i(u)) \leq t(p^{k-1-i}(u))$ since either $p^i(u) \preceq p^{k-1-i}(u)$ or $p^{k-1-i}(u) \preceq p^i(u)$. Therefore $f_i(u)$ is always a valid closed interval on the real line.

Construction of I':
$V(I') = V(T)$ and $E(I') = \{(u, v) \mid f'(u) \cap f'(v) \neq \emptyset\}$ where f' is defined as:

$$f'(u) = [d_T(r, u), d_T(r, u) + k]$$

Lemma 7. *For $0 \leq i \leq k-1$, I_i is a supergraph of T^k.*

Proof. Let $(u,v) \in E(T^k)$. We will show that $(u,v) \in E(I_i)$. Let P be the path between u and v in T. Since $(u,v) \in E(T^k)$, $||P|| \leq k$. It is easy to see that there is exactly one vertex x on P such that $x \preceq u$ and $x \preceq v$. Note that x is the least common ancestor of u and v. Let $d_1 = ||uPx||$ and $d_2 = ||vPx||$. Thus, $x = p^{d_1}(u) = p^{d_2}(v)$ and $||P|| = d_1 + d_2 \leq k$.

Let us assume without loss of generality that $s(p^i(u)) \leq s(p^i(v))$.

If $i \geq d_2$, then $p^i(v) \preceq x \preceq u$ and by Lemma 5, $s(p^i(v)) \leq t(u)$ and also by Lemma 5, $t(u) \leq t(p^{k-1-i}(u))$ implying that $s(p^i(v)) \leq t(p^{k-1-i}(u))$. We now have $s(p^i(u)) \leq s(p^i(v)) \leq t(p^{k-1-i}(u))$. Thus, $f_i(u) \cap f_i(v) \neq \emptyset$ and therefore, $(u,v) \in E(I_i)$.

Now, if $i < d_2$, we have $k-1-i \geq d_1$. Therefore, $p^{k-1-i}(u) \preceq x \preceq v$ and by Lemma 5, $t(v) \leq t(p^{k-1-i}(u))$ and again by Lemma 5, $s(p^i(v)) \leq t(v)$ and so we have $s(p^i(v)) \leq t(p^{k-1-i}(u))$. This means that $s(p^i(u)) \leq s(p^i(v)) \leq t(p^{k-1-i}(u))$. Thus, $f_i(u) \cap f_i(v) \neq \emptyset$ implying that $(u,v) \in E(I_i)$. □

Lemma 8. *I' is a supergraph of T^k.*

Proof. Let $(u,v) \in E(T^k)$. We have to show that $(u,v) \in E(I')$. Let $P = uTv$ and let x be the vertex on P such that $x \preceq u$ and $x \preceq v$ (i.e., x is the least common ancestor of u and v). Let $d_1 = ||uPx||$, $d_2 = ||vPx||$ and $d_3 = ||rTx||$. We have $d_T(r,u) = d_3 + d_1$ and $d_T(r,v) = d_3 + d_2$. Also, since $(u,v) \in E(T^k)$, $d_1 + d_2 \leq k$. Therefore, $|d_1 - d_2| \leq k$ which means that $|d_T(r,u) - d_T(r,v)| \leq k$. Thus, we have $f'(u) \cap f'(v) \neq \emptyset$ implying that $(u,v) \in E(I')$. □

Lemma 9. *If $(u,v) \notin E(T^k)$, then either $(u,v) \notin E(I')$ or $\exists i$ such that $(u,v) \notin E(I_i)$.*

Proof. Let $(u,v) \notin E(T^k)$. Let $P = uTv$ and again let x be the least common ancestor of u and v, i.e., x is the vertex on P such that $x \preceq u$ and $x \preceq v$. Define $d_1 = ||uPx||$ and $d_2 = ||vPx||$; thus, $x = p^{d_1}(u) = p^{d_2}(v)$. Since $(u,v) \notin E(T^k)$, we have $d_1 + d_2 > k$.

Case (i). $d_1 \neq 0$ and $d_2 \neq 0$.

Let us assume without loss of generality that $s(p^{d_1-1}(u)) \leq s(p^{d_2-1}(v))$ By the definition of d_1 and d_2, we have $p^{d_1-1}(u) \npreceq p^{d_2-1}(v)$ and $p^{d_2-1}(v) \npreceq p^{d_1-1}(u))$. Then by Lemma 6, $t(p^{d_1-1}(u)) < s(p^{d_2-1}(v))$. Now applying Lemma 5, we get

$$\text{for any } i,j \text{ such that } 0 \leq i < d_1, 0 \leq j < d_2, t(p^i(u)) < s(p^j(v)) \tag{1}$$

If $1 \leq d_2 \leq k$, consider the interval graph I_j where $j = d_2 - 1$. Now, let $i = k-1-j = k-d_2 < d_1$. Now, from (1), we get $t(p^i(u)) < s(p^j(v))$, that is to say $t(p^{k-1-j}(u)) < s(p^j(v))$. Thus, $f_j(u) \cap f_j(v) = \emptyset$ which means that $(u,v) \notin E(I_j)$.

If $d_2 > k$, then consider I_{k-1}. From (1), we have $t(p^0(u)) < s(p^{k-1}(v))$, and therefore $f_{k-1}(u) \cap f_{k-1}(v) = \emptyset$. Thus, $(u,v) \notin E(I_{k-1})$.

Case (ii). $d_1 = 0$ or $d_2 = 0$.

Now, if $d_1 = 0$, then $u = x \preceq v$ and $d_2 > k$. This implies that $d_T(r, v) > d_T(r, u) + k$. Similarly, if $d_2 = 0$, then $v = x \preceq u$ and $d_1 > k$ implying that $d_T(r, u) > d_T(r, v) + k$. In either case, $f'(u) \cap f'(v) = \emptyset$, and so $(u, v) \notin E(I')$. □

Theorem 1. *For any tree T,* $\text{box}(T^k) \leq k + 1$*, for $k \geq 1$.*

Proof. Let $I', I_0, \ldots, I_{k-1}$ be the interval graphs constructed as explained above. Lemmas 7, 8 and 9 suffice to show that $T^k = I' \cap I_0 \cap \cdots \cap I_{k-1}$. Thus, by Lemma 2, we have the theorem. ■

Corollary 1. *If G is a k-leaf power,* $\text{box}(G) \leq k - 1$*, for $k \geq 2$.*

Proof. It is easy to see that 2-leaf powers are collections of disjoint cliques and thus have boxicity 1. Thus, the corollary is true for $k = 2$. For $k \geq 3$, the statement of the corollary can be proved as follows. From Lemma 3, we have $\text{box}(G) = \text{box}(CC(G))$. From Lemma 4, $CC(G)$ has a $(k-2)$-Steiner root, say T. Now, it follows that $\text{box}(G) = \text{box}(CC(G)) \leq \text{box}(T^{k-2}) \leq k - 1$.

■

3.2 Tightness of the bound

Let the function $w : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ be defined recursively as follows:
$w(1) = 1$, $w(2) = 3$ and for any $i \geq 3$,

$$w(i) = 2(i-1) + 1 + \left[\binom{i-1}{2} \cdot 4 \cdot (w(i-2) - 1) + 1 \right]$$

For any $k \in \mathbb{N}$ and $k \geq 1$, let S_k be the tree shown in figure 1.

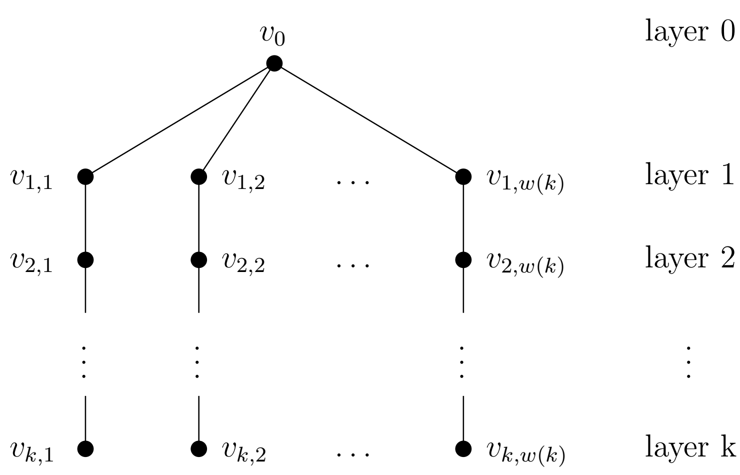

Fig. 1. Tree S_k

Lemma 10. $\mathrm{box}\left((S_k)^k\right) > k-1$.

Proof. Let us prove this using induction on k. It is easy to see that $\mathrm{box}\left((S_1)^1\right) > 0$ and $\mathrm{box}\left((S_2)^2\right) > 1$ (in $(S_2)^2$ vertices $v_{2,1}, v_{2,2}$ and $v_{2,3}$ form an asteroidal triple and therefore by Lemma 1, $(S_2)^2$ is not an interval graph). Let $m \geq 3$ be a positive integer and assume that the statement of the lemma is true for any $k \leq m-1$. We shall now prove by contradiction that $\mathrm{box}\left((S_m)^m\right) > m-1$. For ease of notation, let $S = S_m$. If $\mathrm{box}(S^m) \leq m-1$, then by Lemma 2, there exist $m-1$ interval graphs $I_1, I_2, \ldots, I_{m-1}$ such that $S^m = I_1 \cap \cdots \cap I_{m-1}$. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_{m-1}\}$. For each interval graph I_p, choose an interval representation $\mathcal{R}_p$. For any $u \in V(S^m)$ and $I_p \in \mathcal{I}$, let $left(u, I_p)$ ($right(u, I_p)$) denote the left (right) endpoint of its interval in $\mathcal{R}_p$. We define $L_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,w(m)}\}$, i.e. the set of all vertices in the i-th layer of S^m. Let $interval(u, I_p)$ denote $[left(u, I_p), right(u, I_p)]$, the interval that corresponds to the vertex u in $\mathcal{R}_p$. Note that, since $m \geq 3$, the vertices in layer 1 of S^m form a clique. Therefore, by Helly property, in the interval representation $\mathcal{R}_p$ of each interval graph I_p, the intervals corresponding to the vertices of layer 1 have a common intersection region. Let y_p and z_p denote the left and right endpoints respectively of this common intersection region in $\mathcal{R}_p$. That is, $[y_p, z_p] = \bigcap_{j=1}^{w(m)} interval(v_{1,j}, I_p)$.

Since a vertex in L_m, say $v_{m,j}$, is not adjacent to any vertex $v_{1,j'}$ in layer 1, for $j' \neq j$, there exists at least one interval graph I_p such that $interval(v_{m,j}, I_p)$ is disjoint from the abovementioned common intersection region $[y_p, z_p]$. Define $F(v_{m,j}) = \{I_p \in \mathcal{I} \mid interval(v_{m,j}, I_p) \cap [y_p, z_p] = \emptyset\}$, i.e., the collection of all interval graphs in which $v_{m,j}$ is not adjacent to at least one vertex in layer 1.

Also define $Q(I_p) = \{v_{m,j} \in L_m \mid I_p \in F(v_{m,j})\}$, i.e., the set of all vertices in layer m whose intervals are disjoint from $[y_p, z_p]$ in $\mathcal{R}_p$. Let us partition $Q(I_p)$ into two sets $Q_l(I_p)$ and $Q_r(I_p)$.

$$Q_l(I_p) = \{v_{m,j} \in Q(I_p) \mid left(v_{m,j}, I_p) \leq right(v_{m,j}, I_p) < y_p \leq z_p\}$$

$$Q_r(I_p) = \{v_{m,j} \in Q(I_p) \mid y_p \leq z_p < left(v_{m,j}, I_p) \leq right(v_{m,j}, I_p)\}$$

Partition L_m into two sets A and B such that $A = \{v_{m,j} \mid |F(v_{m,j})| = 1\}$ and $B = \{v_{m,j} \mid |F(v_{m,j})| > 1\}$. Since $|A| + |B| = |L_m| = w(m) = 2(m-1) + 1 + \left[\binom{m-1}{2} \cdot 4 \cdot (w(m-2)-1) + 1\right]$, we encounter at least one of the following two cases. We will show that both the cases lead to contradictions.

Case (i). $|A| \geq 2(m-1)+1$.

Let us partition A into sets $A_1, A_2, \ldots, A_{m-1}$ where $A_i = \{u \in A \mid F(u) = \{I_i\}\}$. Since $|A| \geq 2(m-1)+1$, there exists an A_p with $|A_p| \geq 3$. For a vertex $u \in A_p$, $interval(u, I_p)$ can be either to the left or to the right of $[y_p, z_p]$ in $\mathcal{R}_p$. Thus A_p can be further partitioned into A_p^l and A_p^r where $A_p^l = A_p \cap Q_l(I_p)$ and $A_p^r = A_p \cap Q_r(I_p)$. Since $|A_p| \geq 3$, we have $|A_p^l| \geq 2$ or $|A_p^r| \geq 2$. Without loss of generality, let $|A_p^l| \geq 2$ with $v_{m,j}, v_{m,j'} \in A_p^l$. Also assume without loss of generality that $right(v_{m,j}, I_p) \leq right(v_{m,j'}, I_p) < y_p$. Since $v_{1,j}$ is adjacent to $v_{m,j}$, we have $interval(v_{1,j}, I_p) \cap interval(v_{m,j}, I_p) \neq \emptyset$. Also, by the definition of

$[y_p, z_p]$, $interval(v_{1,j}, I_p) \cap [y_p, z_p] \neq \emptyset$. Therefore, $interval(v_{1,j}, I_p)$ contains both the points $right(v_{m,j}, I_p)$ and y_p, implying that it also contains $right(v_{m,j'}, I_p)$. Thus, $(v_{1,j}, v_{m,j'}) \in E(I_p)$. Since $F(v_{m,j'}) = \{I_p\}$, we know that for all $p' \neq p$, $interval(v_{m,j'}, I_{p'}) \cap [y_{p'}, z_{p'}] \neq \emptyset$ and therefore $(v_{1,j}, v_{m,j'}) \in E(I_{p'})$. This implies that $(v_{1,j}, v_{m,j'}) \in E(I_1 \cap \ldots \cap I_{m-1})$, a contradiction.

Case (ii). $|B| \geq \left[\binom{m-1}{2} \cdot 4 \cdot (w(m-2)-1)\right] + 1$.

For $u \in B$, let $g(u) = \min_{I_i \in F(u)}\{i\}$ and let $g'(u) = \min_{I_i \in F(u) - \{I_{g(u)}\}}\{i\}$. Define $X(u) = \{g(u), g'(u)\}$. Note that both $g(u)$ and $g'(u)$ exists since $u \in B$ and thus $|F(u)| \geq 2$. Let $B_{ij} = \{u \in B \mid X(u) = \{i,j\}\}$. Thus $\mathcal{P} = \{B_{ij} \mid \{i,j\} \subseteq \{1, \ldots, m-1\}\}$ is a partition of B into $\binom{m-1}{2}$ sets. Since $|B| \geq \left[\binom{m-1}{2} \cdot 4 \cdot (w(m-2)-1)\right] + 1$, there exists $B_{pq} \in \mathcal{P}$ such that $|B_{pq}| \geq 4 \cdot (w(m-2)-1) + 1$. Now we partition B_{pq} into 4 sets namely,

$$\begin{aligned}
B_{pq}^{ll} &= B_{pq} \cap Q_l(I_p) \cap Q_l(I_q) \\
B_{pq}^{lr} &= B_{pq} \cap Q_l(I_p) \cap Q_r(I_q) \\
B_{pq}^{rl} &= B_{pq} \cap Q_r(I_p) \cap Q_l(I_q) \\
B_{pq}^{rr} &= B_{pq} \cap Q_r(I_p) \cap Q_r(I_q)
\end{aligned}$$

Since $|B_{pq}| \geq 4 \cdot (w(m-2)-1)+1$, one of these 4 sets will have cardinality at least $w(m-2)$. Let this set be B_{pq}^{lr} (the proof is similar for all the other cases). Thus B_{pq}^{lr} contains $w(m-2)$ vertices, which we will assume without loss of generality to be $v_{m,1}, \ldots, v_{m,w(m-2)}$. Note that for any $v_{m,j} \in B_{pq}^{lr}$, $right(v_{m,j}, I_p) < y_p$ and $z_q < left(v_{m,j}, I_q)$. Let $Y = \{v_{i,j} \mid 2 \leq i \leq m-1, 1 \leq j \leq w(m-2)\}$. Now, since in I_p any vertex $v_{i,j}$ in Y is adjacent to both $v_{m,j}$ and to all the vertices of layer 1, we have $interval(v_{i,j}, I_p) \cap interval(v_{m,j}, I_p) \neq \emptyset$ and $interval(v_{i,j}, I_p) \cap [y_p, z_p] \neq \emptyset$. Since $right(v_{m,j}, I_p) < y_p$, $interval(v_{i,j}, I_p)$ contains the point y_p. Similarly, $interval(v_{i,j}, I_q)$ contains the point z_q. Thus, Y induces a clique in both I_p and I_q. Since v_0 is a universal vertex in S^m, $\{v_0\} \cup Y$ also induces a clique in both I_p and I_q. We claim that in S^m, the induced subgraph of $\{v_0\} \cup Y$ is isomorphic to $(S_{m-2})^{m-2}$. To see this, let $V((S_{m-2})^{m-2}) = \{\overline{v}_0, \overline{v}_{1,1}, \ldots, \overline{v}_{1,w(m-2)}, \overline{v}_{2,1}, \ldots, \overline{v}_{2,w(m-2)}, \ldots, \overline{v}_{m-2,1}, \ldots, \overline{v}_{m-2,w(m-2)}\}$. The isomorphism is given by the bijection $f : \{v_0\} \cup Y \to V((S_{m-2})^{m-2})$ where $f(v_0) = \overline{v}_0$ and $f(v_{i,j}) = \overline{v}_{i-1,j}$. It can be easily verified that f is an isomorphism from the graph induced in S^m by $\{v_0\} \cup Y$ to $(S_{m-2})^{m-2}$. Let

$$G' = \bigcap_{I_i \in \mathcal{I} \setminus \{I_p, I_q\}} I_i$$

Since $\{v_0\} \cup Y$ induced a clique in I_p and I_q, the induced subgraph on $\{v_0\} \cup Y$ in G' is the same as the induced subgraph on $\{v_0\} \cup Y$ in S^m, i.e., $(S_{m-2})^{m-2}$ is an induced subgraph of G'. Therefore, $\mathrm{box}((S_{m-2})^{m-2}) \leq \mathrm{box}(G') \leq m-3$ (from Lemma 2). But this contradicts the induction hypothesis. $\square$

We now construct a tree T_k (see figure 2), for any $k \in \mathbb{N}$ and $k \geq 1$. Define $f(k) = 2k \cdot (w(k)-1) + 1$.

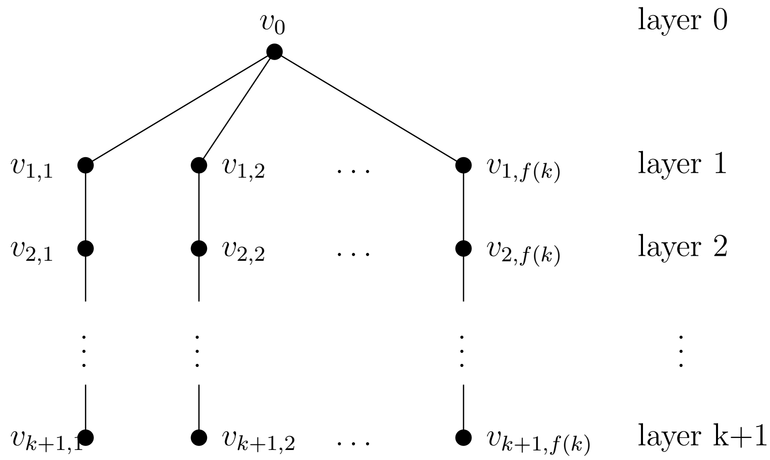

Fig. 2. Tree T_k

Lemma 11. box $\left((T_k)^k\right) > k$.

Proof. We prove this by contradiction. Again, for ease of notation, let $T = T_k$. Assume that box$(T^k) \leq k$. By Lemma 2, there exists a collection of k interval graphs $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ such that $T^k = \bigcap_{I \in \mathcal{I}} I$. Now for each interval graph I_p, for $1 \leq p \leq k$, choose an interval representation $\mathcal{R}_p$. For a vertex $u \in V(T^k)$, let $left(u, I_p)$ ($right(u, I_p)$) denote left(right) endpoint of its interval in $\mathcal{R}_p$. Let $L_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,f(k)}\}$ be the set of all vertices in the i-th layer of T.

For each vertex $v_{k+1,j} \in L_{k+1}$, since $(v_{k+1,j}, v_0) \notin E(T^k)$, there exists at least one interval graph I_p in which $interval(v_{k+1,j}, I_p) \cap interval(v_0, I_p) = \emptyset$. For each interval graph Ip, we define $Q(I_p) = \{v_{k+1,j} \in L_{k+1} \mid interval(v_{k+1,j}, I_p) \cap interval(v_0, I_p) = \emptyset$ and $v_{k+1,j} \notin Q(I_{p'})$ for any $p' < p\}$. Note that $\{Q(I_1), \ldots, Q(I_k)\}$ is a partition of L_{k+1}. We define a partition of $Q(I_p)$ into two sets $Q_l(I_p)$ and $Q_r(I_p)$ as follows. For any vertex $u \in Q(I_p)$, u is in $Q_l(I_p)$ if the interval corresponding to u is to the left of the interval corresponding to v_0 in $\mathcal{R}_p$, otherwise it is in $Q_r(I_p)$. That is,

$$Q_l(I_p) = \{u \in Q(I_p) \mid left(u, I_p) \leq right(u, I_p) < left(v_0, I_p) \leq right(v_0, I_p)\}$$

$$Q_r(I_p) = \{u \in Q(I_p) \mid left(v_0, I_p) \leq right(v_0, I_p) < left(u, I_p) \leq right(u, I_p)\}$$

Now, $\{Q_l(I_i), Q_r(I_i) \mid 1 \leq i \leq k\}$ is a partition of L_{k+1} into $2k$ sets. Since $|L_{k+1}| = f(k) = 2k \cdot (w(k) - 1) + 1$, there exists some set in this partition with size at least $w(k)$. Let us assume this set to be $Q_l(I_p)$ for some p. The proof is similar if the set is $Q_r(I_p)$ and therefore will not be detailed here. Now, we have $|Q_l(I_p)| \geq w(k)$. Let us assume without loss of generality that $v_{k+1,1}, v_{k+1,2}, \ldots, v_{k+1,w(k)} \in Q_l(I_p)$. Let $Y = \{v_{i,j} \mid 1 \leq i \leq k$ and $1 \leq j \leq w(k)\}$. Note that any $v_{i,j} \in Y$ is adjacent to vertices $v_{k+1,j}$ and v_0 in

T^k and therefore also in I_p. Thus, $interval(v_{i,j}, I_p) \cap interval(v_{k+1,j}, I_p) \neq \emptyset$ and $interval(v_{i,j}, I_p) \cap interval(v_0, I_p) \neq \emptyset$. Now, from the definition of $Q_l(I_p)$, it is easy to see that $left(v_0, I_p) \in interval(v_{i,j}, I_p)$ for any $v_{i,j} \in Y$. This means that the vertices in $\{v_0\} \cup Y$ induce a clique in I_p.

It is easy to see that in T^k, the subgraph induced by $\{v_0\} \cup Y$ is isomorphic to $(S_k)^k$. Let

$$G' = \bigcap_{I_i \in \mathcal{I} \setminus \{I_p\}} I_i$$

Since the induced subgraph on $\{v_0\} \cup Y$ in I_p is a clique, the subgraph induced by $\{v_0\} \cup Y$ in G' is the same as the subgraph induced by $\{v_0\} \cup Y$ in T^k, i.e., $(S_k)^k$ is an induced subgraph of G'. Therefore, $\text{box}((S_k)^k) \leq \text{box}(G') \leq k-1$ (from Lemma 2). But this contradicts Lemma 10. □

Hence we have the following theorem.

Theorem 2. *For every $k \in \mathbb{N}$ and $k \geq 1$, $\exists$ a tree τ such that* $\text{box}(\tau^k) > k$.

Corollary 2. *For every $k \in \mathbb{N}$ and $k \geq 2$, $\exists$ a k-leaf power G such that* $\text{box}(G) = k-1$.

Proof. For $k = 2$, any k-leaf power is a collection of disjoint cliques and thus has boxicity 1. The proof for the case when $k \geq 3$ is as follows. Let $G = (T_{k-2})^{k-2}$. Therefore, G is a $(k-2)$-Steiner power (in fact T_{k-2} is a Steiner root for G with no Steiner vertices). Since $CC(G)$ and G are the same graph (note that no two vertices in G have the same neighbourhood), from Lemma 4, G is a k-leaf power. Now, Lemma 11 implies that $\text{box}(G) > k-2$. Using corollary 1, we have $\text{box}(G) = k-1$. ■

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